§6.4 The Gram-Schmidt Process

We've seen that we can do a lot with vectors in a subspace given an orthogonal basis. What if we're not provided with an orthogonal basis and only have a regular basis for our subspace.

Is there a way to obtain an orthogonal basis from any basis?

The answer is yes and relies on orthogonal projections.

Example (2-dimensional case)

Suppose $B = \{x_1, x_2\}$ is a basis for W and we'd like an orthogonal basis $\{v_1, v_2\}$ for W.

We need

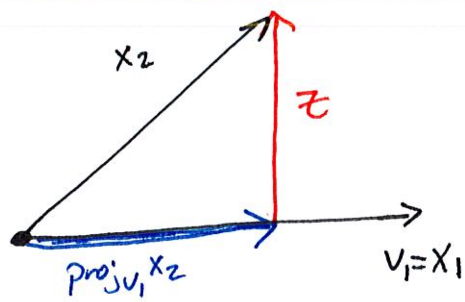
1) $\text{span}\{v_1, v_2\} = \text{span}\{x_1, x_2\} = W$

2) $v_1 \cdot v_2 = 0$ (recall orthogonal sets are linearly independent)

Pick $v_1 = x_1$ (just need a starting point). By the orthogonal projection theorem we can write

$$x_2 = \text{proj}_{v_1} x_2 + z$$

where z is orthogonal to v_1. This is exactly what we want!



we'll take $v_2 = z$. Recall

$$\text{proj}_{v_1} x_2 = \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1} \right) v_1$$

and

$$z = x_2 - \text{proj}_{v_1} x_2$$

Thus

$$v_1 = x_1$$

$$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1} \right) v_1$$

and $\{v_1, v_2\}$ is indeed an orthogonal basis.

How would this work with a basis of more than two vectors?

Gram-Schmidt Algorithm

Let $\{x_1, \dots, x_m\}$ be a basis for a subspace W of $\mathbb{R}^n$. Define

- $v_1 = x_1$
- $v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$
- $v_3 = x_3 - \left(\frac{x_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{x_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2$

$$\vdots \qquad\qquad \vdots$$

- $v_m = x_m - \left(\frac{x_m \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{x_m \cdot v_2}{v_2 \cdot v_2}\right) v_2 - \cdots\cdots - \left(\frac{x_m \cdot v_{m-1}}{v_{m-1} \cdot v_{m-1}}\right) v_{m-1}$

Then $\{v_1, \dots, v_m\}$ is an orthogonal basis for W.

Notice this also be used to produce an orthonormal basis by normalizing each resulting vector.

Example

Let $W = \text{span}\left\{ \underbrace{\begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \end{bmatrix}}_{X_1}, \underbrace{\begin{bmatrix} 5 \\ -6 \\ -1 \\ -5 \end{bmatrix}}_{X_2}, \underbrace{\begin{bmatrix} 5 \\ 2 \\ 6 \\ -3 \end{bmatrix}}_{X_3} \right\}$

a) Find an orthogonal basis for W.

b) Find an orthonormal basis for W.

Solution:

a) • $V_1 = X_1 = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

• $$V_2 = X_2 - \left(\frac{X_2 \cdot V_1}{V_1 \cdot V_1}\right) V_1 = \begin{bmatrix} 5 \\ -6 \\ -1 \\ -5 \end{bmatrix} - \underbrace{\left(\frac{-5-18-2-5}{1+9+4+1}\right)}_{\substack{= -30/15 \\ = -2}} \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -6 \\ -1 \\ -5 \end{bmatrix} - (-2) \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \end{bmatrix}$$

• $$V_3 = X_3 - \left(\frac{X_3 \cdot V_1}{V_1 \cdot V_1}\right) V_1 - \left(\frac{X_3 \cdot V_2}{V_2 \cdot V_2}\right) V_2$$

$$= \begin{bmatrix} 5 \\ 2 \\ 6 \\ -3 \end{bmatrix} - \underbrace{\left(\frac{-5+6+12-3}{1+9+4+1} \right)}_{\substack{= 10/15 \\ = 2/3}} \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \end{bmatrix} - \underbrace{\left(\frac{15+0+18+9}{9+9+9} \right)}_{\substack{= 42/27 \\ = 14/9}} \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 2 \\ 6 \\ -3 \end{bmatrix} - \left(\frac{2}{3}\right) \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \end{bmatrix} - \left(\frac{14}{9}\right) \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 2 \\ 6 \\ -3 \end{bmatrix} - \begin{bmatrix} -2/3 \\ 2 \\ 4/3 \\ 2/3 \end{bmatrix} - \begin{bmatrix} 14/3 \\ 0 \\ 14/3 \\ -14/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for W.

b) since $\|v_1\| = \sqrt{15}$, $\|v_2\| = \sqrt{27} = 3\sqrt{3}$, and $\|v_3\| = \sqrt{2}$

$\left\{ \begin{bmatrix} -1/\sqrt{15} \\ 3/\sqrt{15} \\ 2/\sqrt{15} \\ 1/\sqrt{15} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\}$ is an orthonormal basis for W.

QR Matrix Factorization

Theorem

If A is an $m \times n$ matrix with linearly independent columns, then we can write $A = QR$ where

- Q is an $m \times n$ matrix whose columns form an orthonormal basis for Col A
- R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

Proof

Let $x_1, \dots, x_n$ denote the columns of A. Since $\{x_1, \dots, x_n\}$ is linearly independent it is a basis for col$A$.

Using Gram-schmidt, construct orthonormal basis $\{u_1, \dots, u_n\}$ for Col A and let $Q = [u_1 | \cdots | u_n]$ and let $R = Q^T A$. This follows as

$$R = I_n \cdot R = (Q^T Q) R = Q^T (QR) = Q^T A$$

Q has orthonormal columns so $Q^T Q = I_n$

Example

Let $A = \begin{bmatrix} -1 & 5 & 5 \\ 3 & -6 & 2 \\ 2 & -1 & 6 \\ 1 & -5 & -3 \end{bmatrix}$. Write $A = QR$ with Q and R as stated before.

Solution

From the previous example we found an orthonormal basis of $\text{Col}\,A$ is

$$\left\{ \begin{bmatrix} -1/\sqrt{15} \\ 3/\sqrt{15} \\ 2/\sqrt{15} \\ 1/\sqrt{15} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

so we may take $Q = \begin{bmatrix} -1/\sqrt{15} & 1/\sqrt{3} & 1/\sqrt{2} \\ 3/\sqrt{15} & 0 & 0 \\ 2/\sqrt{15} & 1/\sqrt{3} & 0 \\ 1/\sqrt{15} & -1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$

Now $R = Q^T A$ so

$$R = \begin{bmatrix} -1/\sqrt{15} & 3/\sqrt{15} & 2/\sqrt{15} & 1/\sqrt{15} \\ 1/\sqrt{3} & 0 & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 5 & 5 \\ 3 & -6 & 2 \\ 2 & -1 & 6 \\ 1 & -5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 15/\sqrt{15} & -30/\sqrt{15} & 14/\sqrt{15} \\ 0 & 9/\sqrt{3} & 8/\sqrt{3} \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{15} & -2\sqrt{15} & 14/\sqrt{15} \\ 0 & 3\sqrt{3} & 8/\sqrt{3} \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

Remark

Notice the diagonal entries of R are exactly the lengths of the orthonormal columns of Q.